

# A Modified ICA Approach for Signal Separation in CMB Maps

R. Vio<sup>1</sup> and P. Andreani<sup>2</sup>

<sup>1</sup> Chip Computers Consulting s.r.l., Viale Don L. Sturzo 82, S.Liberale di Marcon, 30020 Venice, Italy  
e-mail: robertovio@tin.it,

<sup>2</sup> ESO, Karl Schwarzschild strasse 2, 85748 Garching, Germany  
INAF-Osservatorio Astronomico di Trieste, via Tiepolo 11, 34143 Trieste, Italy  
e-mail: pandrean@eso.org

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## ABSTRACT

**Aims.** One of the most challenging and important problem of digital signal processing in Cosmology is the separation of foreground contamination from cosmic microwave background (CMB). This problem becomes even more difficult in situations, as the CMB polarization observations, where the amount of available “a priori” information is limited. In this case, it is necessary to resort to the *blind separation* methods. One important member of this class is represented by the *Independent Components Analysis* (ICA). In its original formulation, this method has various interesting characteristics, but also some limits. One of the most serious is the difficulty to take into account any information available in advance. In particular, ICA is not able to exploit the fact that emission of CMB is the same at all the frequencies of observations. Here, we show how to deal with this question. The connection of the proposed methodology with the *Internal Linear Composition* (ILC) technique is also illustrated.

**Methods.** A modification of the classic ICA approach is presented and its characteristics are analyzed both analytically and by means of numerical experiments.

**Results.** The modified version of ICA appears to provide more stable results and of better quality.

**Key words.** Methods: data analysis – Methods: statistical – Cosmology: cosmic microwave background

## 1. INTRODUCTION

The experimental progresses in the detection of cosmological emissions require a parallel development of data analysis techniques in order to extract the maximum physical information from data. In particular, different emission mechanisms are characterized by markedly distinct underlying physical processes. Data analysis often requires the component separation in order to study the individual characteristics. To achieve such a goal, a link between the branch of signal processing science which characterizes and separates different signals and astrophysics is yet well established, and in many cases, modern signal processing techniques have been imported and applied in an astrophysical context. This is the case of the Independent Component Analysis (ICA), used for the separation of the Cosmic Microwave Background (CMB) from diffuse foregrounds originated by our own Galaxy (see Stivoli et al. 2006, and references therein). This techniques offers several advantages. In particular, under the only assumption of mutual statistical independence, it permits the separation of all the components contributing to an observed signal.

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More formally, let the available data in the form of  $N$  mean-subtracted maps  $\{\mathcal{Y}\}_{i=1}^N$ , corresponding to different observing channels, and containing  $M$  pixels each. If  $\mathbf{Y}_i = \text{VEC}^T[\mathcal{Y}_i]$ <sup>1</sup>, these maps can be arranged in a  $N \times M$  matrix

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \\ \vdots \\ \mathbf{Y}_N \end{pmatrix}. \quad (1)$$

A common assumption in CMB observations is that each  $\mathcal{Y}_i$  is given by the linear mixture (or in astrophysical term “frequency channel”) of  $N_c$  components  $\{\mathcal{S}_j\}_{j=1}^{N_c}$  due to different physical processes (e.g., free-free, dust re-

<sup>1</sup> We recall that the operator  $\text{VEC}[\mathbf{H}]$  transforms a matrix  $\mathbf{H}$  into a vector by stacking its columns one underneath the other. Hence,  $\text{VEC}^T[\mathbf{H}]$  provides a row array.

radiation, bremsstrahlung, ...). In formula,

$$\mathbf{Y}_i = \sum_{j=1}^{N_c} a_{ij} \mathbf{S}_j, \quad (2)$$

with  $\mathbf{S}_j = \text{VEC}^T[\mathcal{S}_j]$  and  $\{a_{ij}\}$  constant coefficients. With this model, it is hypothesized that for the  $j$ th physical process a template  $\mathcal{S}_j$  exists that is independent of the observing channel “ $i$ ”. Although rather strong, actually it is not unrealistic to assume that this condition is satisfied when small patches of the sky are considered. In matrix notation, Eq. (2) can be written in the form

$$\mathbf{Y} = \mathbf{A}\mathbf{S}, \quad (3)$$

where

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \vdots \\ \mathbf{S}_{N_c} \end{pmatrix}, \quad (4)$$

and

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N_c} \\ a_{21} & a_{22} & \dots & a_{2N_c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN_c} \end{pmatrix}, \quad (5)$$

denotes the so called *mixing matrix*.

Here the problem is that only the mixtures  $\mathbf{Y}$  are available, whereas neither  $\mathbf{A}$  nor  $\mathbf{S}$  are known. Hence, the issue raises whether from  $\mathbf{Y}$  it is possible to obtain the components  $\{\mathbf{S}_i\}$ . Surprisingly, a positive answer is possible. At this point, however, it is necessary to stress that the problem presents a basic ambiguity. In particular, at best each  $\mathbf{S}_i$  can be determined unless a multiplicative constant. In fact, if  $\mathbf{S}_i$  is multiplied by a scalar and the corresponding  $i$ -th column of  $\mathbf{A}$  is divided by same quantity, then an identical model is obtained. For this reason, it is customary to assume that the variance of  $\mathbf{S}_i$  is equal to one, i.e.,  $\mathbf{S}_i \mathbf{S}_i^T / M = 1$ .

For simplicity, in the following it is assumed that  $N_c = N$ , i.e. that the number of observed mixtures is equal to the number of components. In this way,  $\mathbf{A}$  is a square matrix.

## 2. A CLASSIC APPROACH: ICA

One of the most celebrated technique for the blind separation of signals in mixtures is the so called *independent component analysis* (ICA). The basic idea behind ICA is rather simple (and obvious): to obtain the separation of the components it is sufficient to have an estimate of  $\mathbf{A}^{-1}$ . In fact,  $\mathbf{S} = \mathbf{A}^{-1}\mathbf{Y}$ . Now, if the CMB component and the Galactic ones are mutually uncorrelated, i.e. if the corresponding covariance matrix is given by  $\mathbf{C}_S = \mathbf{S}\mathbf{S}^T / M = \mathbf{I}$ , then

$$\mathbf{Y}\mathbf{Y}^T / M = \mathbf{A}\mathbf{A}^T. \quad (6)$$

This system of equations defines  $\mathbf{A}$  but an orthogonal matrix. In fact, if  $\mathbf{A} = \mathbf{Z}\mathbf{V}$ , with  $\mathbf{V}$  orthogonal, then  $\mathbf{Y}\mathbf{Y}^T / M = \mathbf{A}\mathbf{A}^T = \mathbf{Z}\mathbf{V}\mathbf{V}^T\mathbf{Z}^T = \mathbf{Z}\mathbf{Z}^T$ . The problem is that, given the symmetry of  $\mathbf{Y}\mathbf{Y}^T$ , system (6) contains only  $N(N+1)/2$  independent equations, but the estimates of  $N^2$  quantities should be necessary. In ICA, the  $N(N-1)/2$  remaining equations are obtained by enforcing the constraint that the components  $\{\mathbf{S}_i\}$  are not only mutually uncorrelated but also mutually independent. In other words, the separation problem is converted into the form <sup>2</sup>

$$\hat{\mathbf{S}} = \arg \min_{\mathbf{S}} \mathbf{F}(\mathbf{S}), \quad (7)$$

$$\text{subject to } \hat{\mathbf{S}} = \mathbf{A}^{-1}\mathbf{Y} \quad \text{and} \quad \mathbf{Y}\mathbf{Y}^T / M = \mathbf{A}\mathbf{A}^T, \quad (8)$$

with  $\mathbf{F}(\mathbf{S})$  a function that measures the independence between the components  $\{\mathbf{S}_i\}$ . The definition of a reliable measure  $\mathbf{F}(\cdot)$  is not a trivial task. In literature, various choices are available (see Hyvärinen et al. 2001, and reference therein). In practical algorithms, the optimization problem is not implemented explicitly in the form (7)-(8). Typically, a first estimate  $\hat{\mathbf{S}}_*$  is obtained through a *principal component analysis* (PCA) step followed by a *spherizing* operation (i.e. forcing  $\hat{\mathbf{S}}_* \hat{\mathbf{S}}_*^T / M = 1$ ). In this way, a set of uncorrelated components become available with  $\mathbf{C}_{\hat{\mathbf{S}}_*} = \mathbf{I}$ , as well the corresponding mixing matrix  $\mathbf{A}_*$ . Later, these estimated quantities are iteratively refined to maximize  $\mathbf{F}(\mathbf{S})$  and to get the final  $\hat{\mathbf{S}}$ . Again, in literature, various techniques are available. Among these, one of the most famous and used algorithm is *FASTICA* based on a *fixed-point* optimization approach (Hyvärinen et al. 2001; Maino 2002; Baccigalupi et al. 2004). An alternative technique is *JADE* that makes use of the *joint diagonalization algorithm* (Cardoso 1999).

## 3. A SUBSPACE APPROACH

The main benefit of ICA is the fact that, apart from the independence of the components  $\{\mathbf{S}_i\}$ , it does not make use of further assumptions. If from one side, this makes the method easy to use, on the other one it does not permit the exploitation of any information that be available in advance. In particular, in the case of the CMB observations, it is expected that, at least on small patches of the sky, the mixing matrix  $\mathbf{A}$  can be written in form:

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & \dots & a_{1N} \\ 1 & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_{N2} & \dots & a_{NN} \end{pmatrix}. \quad (9)$$

This means that the component  $\mathbf{S}_1$ , here assumed to correspond to the CMB emission, gives the same contribution in all the observed mixtures. Here, the question is how to

<sup>2</sup> We recall that the function “ $\arg \min F(x)$ ” provides the values of  $x$  of for which the function  $F(x)$  has the smallest value.

implement this piece of information. A possible solution can be obtained if model (3) is written in the form:

$$\mathbf{Y} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \mathbf{S}_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{N2} \end{pmatrix} \mathbf{S}_2 + \cdots + \begin{pmatrix} a_{1N} \\ a_{2N} \\ \vdots \\ a_{NN} \end{pmatrix} \mathbf{S}_N. \quad (10)$$

This equation enlightens the fact that the columns of  $\mathbf{A}$  span a *signal space*  $\langle \mathbf{A} \rangle = \text{span}\{\mathbf{1}, \mathbf{a}_2, \dots, \mathbf{a}_N\}$  where  $\mathbf{1} = (1, 1, \dots, 1)^T$  and  $\mathbf{a}_i$  denotes the  $i$ -th column of  $\mathbf{A}$ . In other words, signal  $\mathbf{Y}$  lives in a  $N$ -dimensional space. Moreover, from the same equation it is evident that if  $\mathbf{Y}$  is projected onto a  $(N-1)$ -dimensional subspace orthogonal to vector  $\mathbf{1}$ , then the contribution of  $\mathbf{S}_1$  is removed from  $\mathbf{Y}$  itself. This job can be done by means of the projection matrix

$$\mathbf{P}_\perp = \mathbf{I} - \mathbf{1}(\mathbf{1}^T \mathbf{1})^{-1} \mathbf{1}^T. \quad (11)$$

The application of ICA to

$$\mathbf{Y}_\perp = \mathbf{P}_\perp \mathbf{Y}, \quad (12)$$

permits to obtain the estimates  $\hat{\mathbf{S}}_2, \hat{\mathbf{S}}_3, \dots, \hat{\mathbf{S}}_N$  of the  $N-1$  components  $\mathbf{S}_2, \mathbf{S}_3, \dots, \mathbf{S}_N$ . Hence, since it is  $\{\hat{\mathbf{S}}_i \hat{\mathbf{S}}_j^T / M\}_{i,j=2}^N = \delta_{ij}$  ( $\delta_{ij}$  denotes the Kronecker function) and by assumption  $\{\hat{\mathbf{S}}_1 \hat{\mathbf{S}}_i^T\}_{i=2}^N = 0$ , the columns of  $\mathbf{A}$  can be estimated through Eq. (10) by means of

$$\hat{\mathbf{a}}_i = \mathbf{Y} \hat{\mathbf{S}}_i^T / M, \quad i = 2, 3, \dots, N. \quad (13)$$

At this point, always using Eq. (10),  $\hat{\mathbf{S}}_1$  can be derived from

$$\hat{\mathbf{S}}_1 = \mathbf{1}^T (\mathbf{Y} - \hat{\mathbf{A}}_{-1} \hat{\mathbf{S}}_{-1}) / N, \quad (14)$$

where

$$\hat{\mathbf{A}}_{-1} = (\hat{\mathbf{a}}_2, \dots, \hat{\mathbf{a}}_N), \quad (15)$$

and

$$\hat{\mathbf{S}}_{-1} = \begin{pmatrix} \hat{\mathbf{S}}_2 \\ \vdots \\ \hat{\mathbf{S}}_N \end{pmatrix}. \quad (16)$$

Since  $\hat{\mathbf{S}}_1 \hat{\mathbf{S}}_i^T = 0$ ,  $i \neq 1$ , it is not difficult to see that

$$\hat{\mathbf{S}}_1 = \mathbf{S}_1. \quad (17)$$

It is worth noticing that, contrary to the other components, the variance of  $\hat{\mathbf{S}}_1$  is not one:  $\hat{\mathbf{S}}_1 \hat{\mathbf{S}}_1^T / M \neq 1$ . This is a consequence of the fact that  $\mathbf{a}_1 \equiv \mathbf{1}$  is fixed.

Here, it is necessary to stress that, if one is interested only in the component  $\mathbf{S}_1$ , then the situation is simpler since it is not necessary that the other components are independent of even uncorrelated. The point is that the separation of the components  $\{\mathbf{S}_i\}_{i=2}^N$  has no effect on the computation of  $\mathbf{S}_1$ . In other words, it does not matter whether these components are correctly disentangled or not. In fact, the same  $\hat{\mathbf{S}}_1$  as in Eq. (14) is obtained if, instead of the independent  $\{\hat{\mathbf{S}}_i\}_{i=2}^N$ , in Eq. (13), the orthogonal  $\{\hat{\mathbf{S}}_i^*\}_{i=2}^N$  are used that are computed through the

application of the PCA to  $\mathbf{Y}_\perp$ . This is because Eq. (13) requires the orthogonality of the components, not their independence. Moreover, Eq. (3) can be written in the equivalent form

$$\mathbf{Y} = \tilde{\mathbf{A}} \tilde{\mathbf{S}}, \quad (18)$$

where

$$\tilde{\mathbf{A}} = \mathbf{A} \mathbf{H}, \quad (19)$$

$$\tilde{\mathbf{S}} = \mathbf{H}^{-1} \mathbf{S}, \quad (20)$$

$$\mathbf{H} = \left( \begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \Phi & \\ 0 & & & \end{array} \right), \quad (21)$$

and  $\Phi$  is any arbitrary  $(N-1) \times (N-1)$  non-singular matrix. Now, since  $\tilde{\mathbf{S}}_1 = \mathbf{S}_1$  and the first column of  $\tilde{\mathbf{A}}$  is still  $\mathbf{1}$ , the meaning of this equation is that there is an infinite number of sets  $\{\tilde{\mathbf{S}}_i\}_{i=2}^N$  that, when projected onto the  $N-1$  subspace orthogonal to  $\mathbf{1}$ , produce the same  $\mathbf{Y}_\perp$ . Hence, for the separation of  $\mathbf{S}_1$  it is not necessary the use of the true  $\{\mathbf{S}_i\}_{i=2}^N$  but only one of such sets.

Of course, these are theoretical results. In practical situations, it is quite improbable that for finite signals the condition  $\{\mathbf{S}_1 \mathbf{S}_i^T\}_i = 0$ ,  $i \neq 1$  be strictly satisfied. In fact, because of the statistical fluctuations, in general the components  $\{\mathbf{S}_i\}_{i=1}^N$  present a certain degree of mutual correlation even in the case they are the realization of independent, stationary, stochastic random processes. Therefore, enforcing the condition  $\hat{\mathbf{S}}_1 \hat{\mathbf{S}}_i = 0$ ,  $i \neq 1$ , makes inaccurate the estimate of the coefficients  $\{\hat{\mathbf{a}}_i\}$  as provided by Eq. (13).

## 4. SOME NUMERICAL EXPERIMENTS

Because of the arguments presented above, it is to be expected that, with respect to the classic ICA, the use of a subspace method provides more accurate and stable results. Given the non-linear nature of the algorithms, the verification of this expectation has to be made through numerical experiments.

Here, non-astronomical subjects have been deliberately chosen. In this way, a direct visualization of the separation is possible and hence an easier and safer assessment of its goodness. Moreover, the use of *deterministic* subjects make easier the modeling of various experimental conditions (e.g., the sample correlation between different images can be obtained using images with almost-constant luminosity areas in correspondence to the same coordinates).

Figures 1-2 show the results provided by ICA and its version based on the subspace approach (modified ICA) when three mixtures are available each containing the contribution of an equal number of (almost) uncorrelated components with a mixing matrix

$$\hat{\mathbf{A}} = \begin{pmatrix} 1.0 & 0.2 & 0.4 \\ 1.0 & 0.5 & 0.3 \\ 1.0 & 0.3 & 0.6 \end{pmatrix}. \quad (22)$$

No noise has been added. The examination of these figures seems to indicate that both techniques are able to produce an excellent separation. Actually, the results provided by the classic ICA are unstable and often unsatisfactory. The point is that this method is non-linear and therefore the algorithms have to be initialized. As a consequence, different results can be obtained according to the chosen initialization. This is quantified in Fig. 3, where the normalized norm of the residuals,  $\|\hat{\mathbf{S}}_i - \mathbf{S}_i\|/\|\mathbf{S}_i\|$ , corresponding to 200 different initializations are shown. The residuals are computed from the difference between the real solution and the estimated one. From this figure, it is clear that the classical ICA provides more more than one solution, while the modified ICA method produces a stable  $\hat{\mathbf{S}}_1$ . This is a consequence of the fact that in the subspace approach such component is computed via linear operations only.

The question of the stability of the solution may be considered of secondary importance. Actually, this is not true. The ICA method does not permit to check if a specific solution is satisfactory or not. Any separation provides components that, when summed up, will perfectly reproduce the original mixtures. In a real experimental situation, this means the unavailability of a reliable selection criterion. For example, a simple selection criterion could be based on the frequency with which a solution is obtained for different initializations of the algorithm. However, there is no guarantee that the most frequent solution represents the best one. This is the case for the experiment in Figs. 4-6 where, contrary to the previous one, the components present a certain degree of correlation with a (normalized) cross-product matrix

$$\hat{\mathbf{C}}_{\mathbf{S}} = \begin{pmatrix} 1.00 & 0.20 & 0.06 \\ 0.20 & 1.00 & 0.06 \\ 0.06 & 0.06 & 1.00 \end{pmatrix}. \quad (23)$$

As expected, the separation is by far less satisfactory than in the previous experiment. However, also in this case the results concerning the subspace method appear more stable. Here, the point of interest is that from the examination of the top-left panel in Fig. 6 it is possible to see that the classic ICA provides the estimate  $\hat{\mathbf{S}}_1$  closest to the true solution. However, apart from the fact that the  $\hat{\mathbf{S}}_2$  corresponding to such estimate is systematically the worst among those obtained, the frequency with which the “best”  $\hat{\mathbf{S}}_1$  is found is very small. In a practical situation, such a solution should have been discarded.

It has not to surprise that, when the conditions of applicability are violated, the classic ICA method is able to “see” a good estimate of  $\mathbf{S}_1$  that is “unreachable” with the subspace approach. This is because, as stated above, the classic ICA works with a larger number of unknowns and therefore is more flexible and can span a wider “solution space”. However, there is no guarantee that such a flexibility is effectively fruitful. The situation is similar (although not identical) to a polynomial fit: the use of high degree functions permits a greater flexibility but at the cost of a remarkable instability of the results that often make quite hard, if not impossible, the choice of a good solution.

## 5. RELATIONSHIP WITH THE ILC METHOD

In CMB literature, another method has been often used for the separation of Cosmic signals from the Galactic foregrounds. This is the so called *internal linear combination* method (ILC) (Bennett et al. 2003; Eriksen et al. 2004; Hinshaw et al. 2007). The aim of this method is not the separation of all the signals that contribute to the observed mixtures, but only the extraction of the specific component  $\mathbf{S}_1$ . With ILC a solution is searched in the form

$$\hat{\mathbf{S}}_1 = \mathbf{w}^T \mathbf{Y}, \quad (24)$$

with the column vector  $\mathbf{w}$  providing a set of appropriate weights. Since, the basic assumption is that  $\mathbf{S}_1$  is the same in all the mixtures, i.e.,

$$\mathbf{Y} = \mathbf{1}\mathbf{S}_1 + \mathcal{N}; \quad (25)$$

with  $\mathcal{N}$  a zero-mean noise that provides the contribution of all the components other than that of interest, it is imposed that

$$\mathbf{1}^T \mathbf{w} = 1. \quad (26)$$

In this way,

$$\hat{\mathbf{S}}_1 = \mathbf{S}_1 + \mathbf{w}^T \mathcal{N}, \quad (27)$$

i.e. the weights do not alter the  $\mathbf{S}_1$  component. For the same assumption, among all the possible solutions provided by Eq. (24) with the condition (26), that of interest has the property that  $\sigma^2 = \hat{\mathbf{S}}_1 \hat{\mathbf{S}}_1^T$  is a minimum. In fact, assuming the noise  $\mathcal{N}$  uncorrelated with  $\mathbf{S}$ , it is

$$\sigma^2 = \frac{1}{M} [\mathbf{S}_1 \mathbf{S}_1^T + \mathbf{w}^T \mathcal{N} \mathcal{N}^T \mathbf{w}]. \quad (28)$$

Hence, the minimization of  $\sigma^2$  with respect to  $\mathbf{w}$  implies the strongest filtering of the component  $\mathcal{N}$ . It can be shown that the weights which minimize this quantity are given by (Eriksen et al. 2004)

$$\mathbf{w} = \frac{\mathbf{C}_{\mathbf{Y}}^{-1} \mathbf{1}}{\mathbf{1}^T \mathbf{C}_{\mathbf{Y}}^{-1} \mathbf{1}}, \quad (29)$$

where  $\mathbf{C}_{\mathbf{Y}} = \mathbf{Y} \mathbf{Y}^T / M$ . Hence, the ILC estimator takes the form

$$\hat{\mathbf{S}}_1 = \frac{\mathbf{1}^T \mathbf{C}_{\mathbf{Y}}^{-1} \mathbf{Y}}{\mathbf{1}^T \mathbf{C}_{\mathbf{Y}}^{-1} \mathbf{1}}. \quad (30)$$

Although this estimator appears different from that given by Eq. (14), actually they provide identical results. In fact, under model (10), it is

$$\mathbf{C}_{\mathbf{Y}} = \mathbf{A} \mathbf{C}_{\mathbf{S}} \mathbf{A}^T. \quad (31)$$

If this equation is inserted in Eq. (30), one obtains

$$\hat{\mathbf{S}}_1 = \alpha \mathbf{1}^T \mathbf{A}^{-T} \mathbf{C}_{\mathbf{S}}^{-1} \mathbf{S}, \quad (32)$$

with the scalar  $\alpha$  given by

$$\alpha = [\mathbf{1}^T \mathbf{A}^{-T} \mathbf{C}_{\mathbf{S}}^{-1} \mathbf{A}^{-1} \mathbf{1}]^{-1}, \quad (33)$$

and  $\mathbf{A}^{-T} \equiv (\mathbf{A}^{-1})^T$ . Now, since it is trivially verified that

$$\mathbf{1}^T = \mathbf{e}_1^T \mathbf{A}^T, \quad (34)$$

where

$$\mathbf{e}_1 \equiv (1, 0, \dots, 0)^T, \quad (35)$$

it is

$$\mathbf{1}^T \mathbf{A}^{-T} = \mathbf{e}_1^T. \quad (36)$$

Hence,  $\alpha = (\mathbf{C}_{\mathbf{S}}^{-1})_{11} = ([\mathbf{S}_1 \mathbf{S}_1^T])^{-1} = \sigma_{11}^{-1}$ . As a consequence, if  $\mathbf{S}_1$  is uncorrelated with  $\{\mathbf{S}_i\}_{i=2}^N$  i.e. if

$$\mathbf{C}_{\mathbf{S}} = \left( \begin{array}{c|ccc} \sigma_{11} & 0 & 0 & \dots & 0 \\ 0 & \sigma_{22} & \sigma_{23} & \dots & \sigma_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \sigma_{N2} & \sigma_{N3} & \dots & \sigma_{NN} \end{array} \right), \quad (37)$$

from Eq. (32) and the fact that  $\mathbf{C}_{\mathbf{S}}^{-1}$  has the form

$$\mathbf{C}_{\mathbf{S}}^{-1} = \left( \begin{array}{c|ccc} \sigma_{11}^{-1} & 0 & 0 & \dots & 0 \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \boldsymbol{\Omega}^{-1} \end{array} \right), \quad (38)$$

with  $\boldsymbol{\Omega}$  the bottom-right block of the matrix in the rhs of Eq. (37), one obtains that

$$\hat{\mathbf{S}}_1 = \mathbf{e}_1^T \mathbf{S} = \mathbf{S}_1, \quad (39)$$

i.e., the same result as Eq. (17). More in general, the estimators (14) and (32) provide identical results also when the components  $\mathbf{S}_1$  is not uncorrelated with the other ones and/or instrumental noise is added to the observed mixtures. The reason is that, as stated earlier, ILC provides an estimate  $\hat{\mathbf{S}}_1$  with the property that the quantity  $\sigma^2$  in Eq. (28) is a minimum. Although not evident from the treatment in Sec. 3, the same holds for the estimator (14). In fact, it is not difficult to realize that, after the determination of the components  $\{\hat{\mathbf{S}}_i\}_{i=2}^N$ , the coefficients  $\{\hat{\mathbf{a}}_i\}_{i=2}^N$ , as given by Eq. (13), are the solution of

$$\frac{d\sigma^2}{d\hat{\mathbf{A}}_{-1}} = \frac{d(\hat{\mathbf{S}}_1 \hat{\mathbf{S}}_1^T)}{d\hat{\mathbf{A}}_{-1}} = 0 \quad (40)$$

with

$$\hat{\mathbf{S}}_1 \hat{\mathbf{S}}_1^T = \mathbf{1}^T (\mathbf{Y} - \hat{\mathbf{A}}_{-1} \hat{\mathbf{S}}_{-1}) (\mathbf{Y} - \hat{\mathbf{A}}_{-1} \hat{\mathbf{S}}_{-1})^T \mathbf{1} \quad (41)$$

that is derived from the sample version of Eq. (10).

## 6. FINAL REMARKS

In the previous section it has been assumed that the number  $N$  of the observed mixtures (or frequency channels, images at different observing frequency) equals the number  $N_c$  of the components. In practical application this coincidence is improbable. Of course, in order the subspace approach can work satisfactorily, it is necessary to know the correct dimension of the signal-space. Therefore, the question raises on what happens when  $N_c \neq N$ . If from

one side, the case  $N < N_c$  does not offer many possibilities to obtain meaningful results, on the other one the case  $N > N_c$  can be successfully addressed. In fact,  $N_c$  can be determined by the number of non-zero eigenvalues of matrix  $\mathbf{C}_{\mathbf{Y}}$  and the corresponding eigenvectors can be used to construct a basis of the signal-space with the correct dimensionality. After that, it is sufficient to project  $\mathbf{Y}$  onto this space, obtaining a “new” set of  $N_c^* < N$  mixtures  $\mathbf{Y}_R$ , and then to work with this. The rest of the procedure in Sect. 3 and Sec. 5 remains the same. More in particular, if the  $N \times N$  matrix  $\mathbf{C}_{\mathbf{Y}}$  is decomposed in the form

$$\mathbf{C}_{\mathbf{Y}} = \mathbf{V} \boldsymbol{\Lambda} \mathbf{V}^T \quad (42)$$

where  $\boldsymbol{\Lambda}$  is a diagonal matrix containing the eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{N_c} > \lambda_{N_c+1} = \dots = \lambda_N = 0$ , whereas  $\mathbf{V}$  is an orthogonal matrix whose columns contain the corresponding eigenvectors, then

$$\mathbf{Y}_R = \boldsymbol{\Lambda}_{\emptyset}^{-1/2} \mathbf{V}^T \mathbf{Y}, \quad (43)$$

where  $\boldsymbol{\Lambda}_{\emptyset}^{-1/2}$  is a diagonal matrix whose non-zero values are given by the inverse of the non-zero entries of  $\boldsymbol{\Lambda}^{1/2}$ .

More complex is the situation when  $\mathbf{Y}$  is contaminated by measurements errors  $\mathbf{N}$ . If  $\mathbf{N}$  is zero-mean and additive, model (3) converts into

$$\mathbf{Y} = \mathbf{A} \mathbf{S} + \mathbf{N}. \quad (44)$$

Here the problem is that the effective number of components becomes  $N_c + N$  and therefore the separation problem is always underdetermined. Therefore, although  $E[\mathbf{N}] = \mathbf{0}$ , it happens that  $E[\hat{\mathbf{S}}_1] \neq \mathbf{S}_1$ . In other words, a bias is present (Vio & Andreani 2008). This point is more evident if the ILC estimator (30) is considered. There, the bias is due to the fact that matrix  $\mathbf{C}_{\mathbf{y}}$  is no longer given by Eq. (31) but by

$$\mathbf{C}_{\mathbf{Y}} = \mathbf{A} \mathbf{C}_{\mathbf{S}} \mathbf{A}^T + \mathbf{C}_{\mathbf{N}}. \quad (45)$$

This equation suggests that a simple way to remove the bias is to use  $\mathbf{C}_{\mathbf{Y}} - \mathbf{C}_{\mathbf{N}}$  instead of  $\mathbf{C}_{\mathbf{Y}}$ . Something similar holds also for  $\hat{\mathbf{S}}_1$  as provided by Eq. (14). In fact, after some algebra it is possible to show that an equivalent form is

$$\hat{\mathbf{S}}_1 = \mathbf{1}^T [\mathbf{I} - \mathbf{C}_{\mathbf{Y}} \mathbf{P}_{\perp}^T (\mathbf{P}_{\perp} \mathbf{C}_{\mathbf{Y}} \mathbf{P}_{\perp}^T)^{\dagger} \mathbf{P}_{\perp}] \mathbf{Y} / N, \quad (46)$$

where symbol “ $\dagger$ ” denotes *pseudo-inverse*. Hence, the same arguments apply as above. Concerning the influence of the noise of the components  $\{\mathbf{S}_i\}$ ,  $i \neq 1$ , the situation is much more difficult since similar to that encountered in the classic ICA approach (for details, see Hyvärinen et al. 2001).

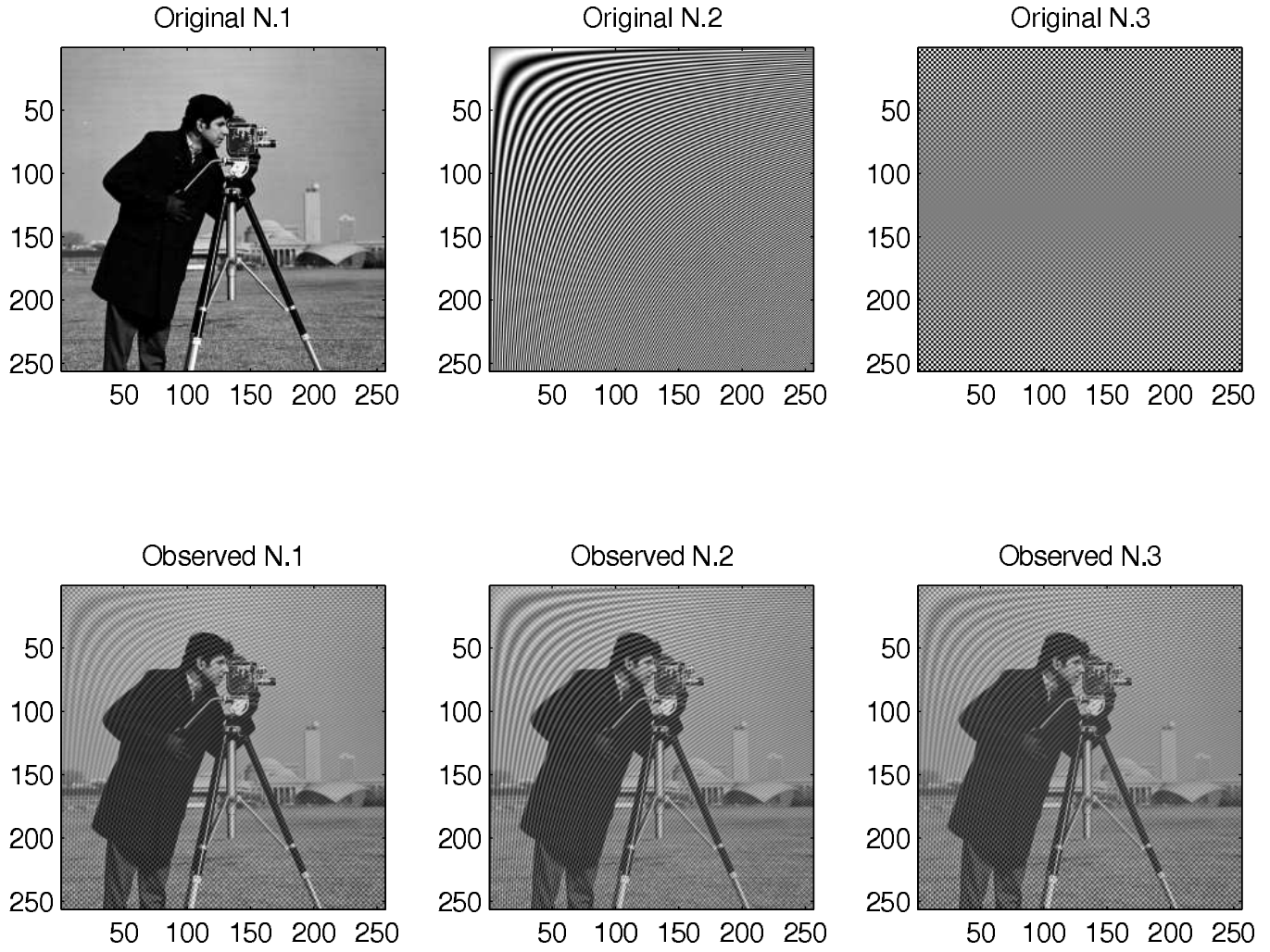
## 7. SUMMARY

In this paper we have considered the problem of a modification of the ICA separation technique that permits to exploit the “*a priori*” information the, contrary to the

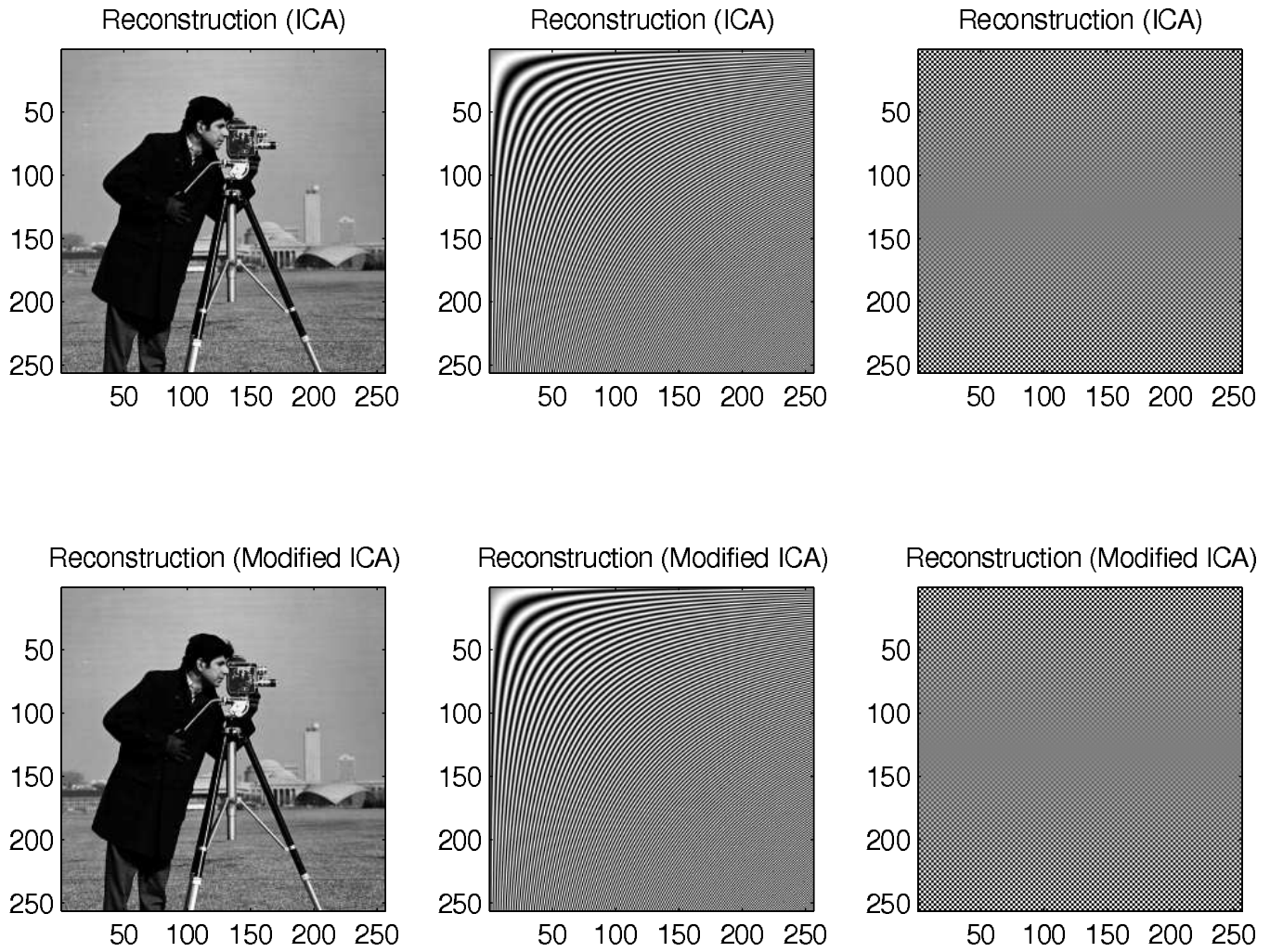
Galactic components, the contribution of CMB to the microwave maps is independent of the observing frequency. A subspace approach has been proposed that is more stable and provide more accurate results than the classic ICA technique. A relationship between this approach and the *Internal Linear Composition* method has been also shown.

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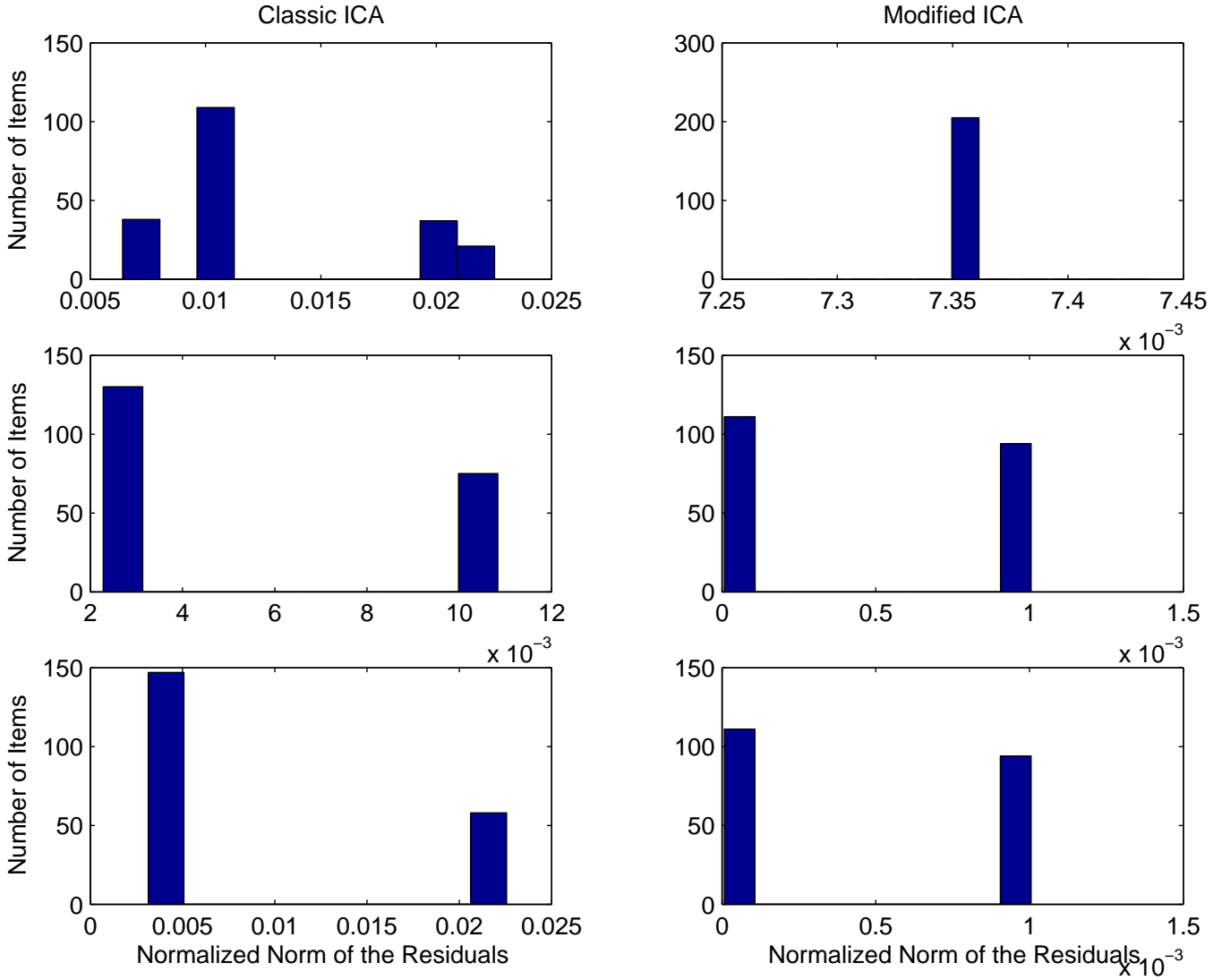


**Fig. 1.** **Top panels** – Original images  $S$  used in the experiment dealing with component separation as described in Sect. 4; **Bottom panels** – Observed mixtures  $Y = AS$ , with  $A$  given by Eq. (22). In this experiment the components are almost uncorrelated.

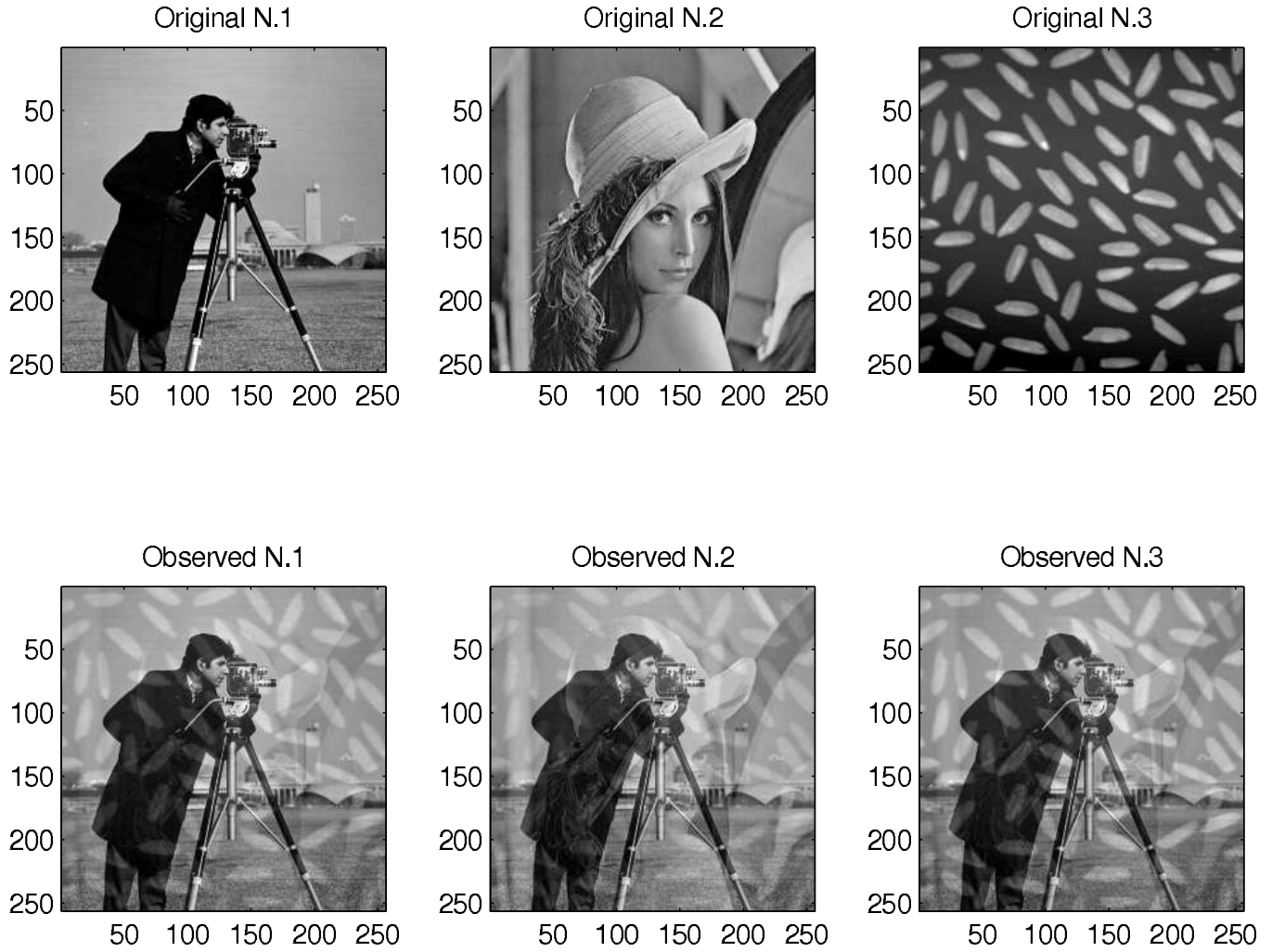


**Fig. 2. Top panels** – Typical separation obtained with the classic ICA algorithm when applied to the observed mixtures shown in Fig. 1; **Bottom panels** – Typical separation obtained with the subspace based ICA when applied to the observed mixtures shown in Fig. 1.

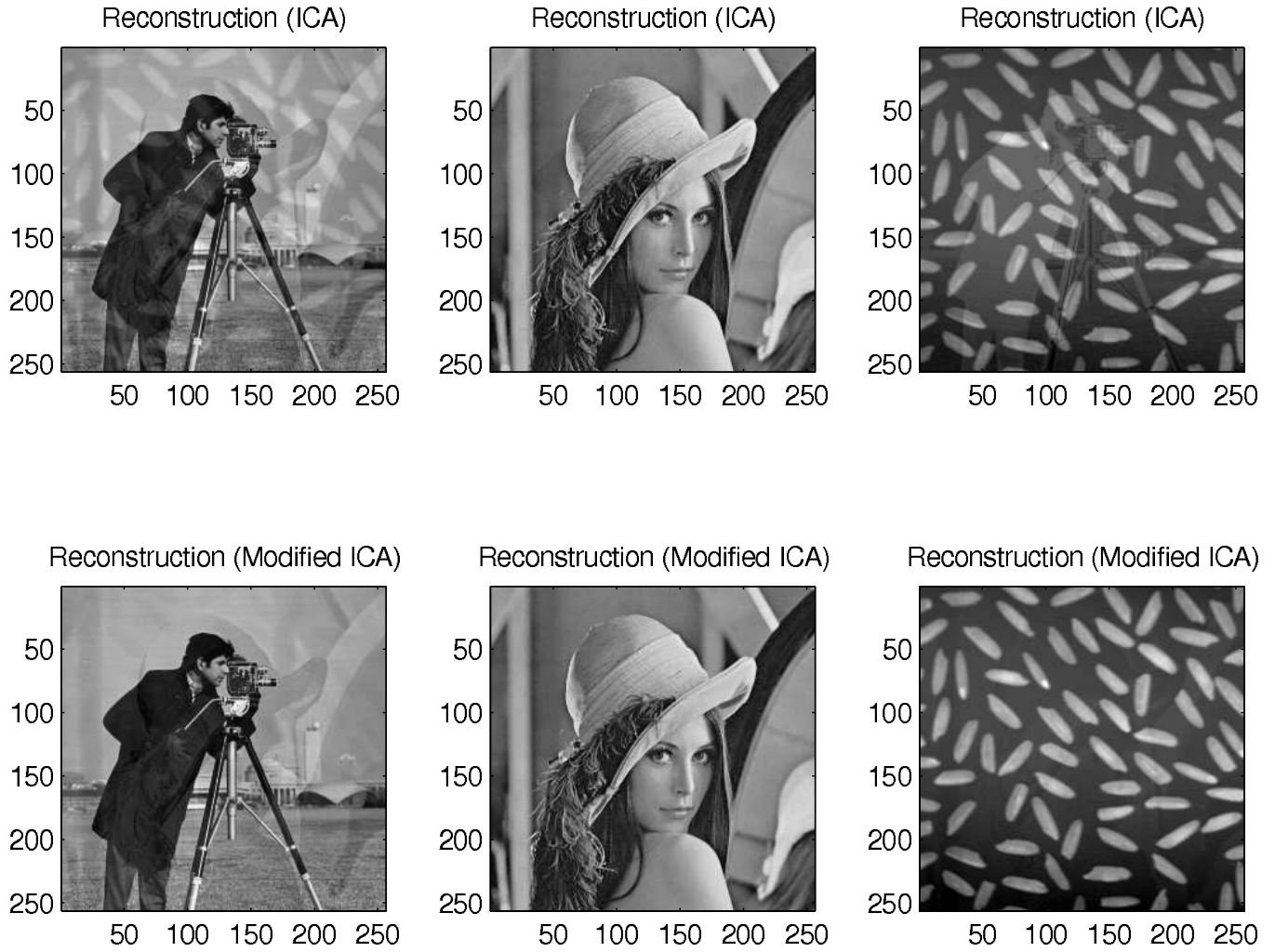




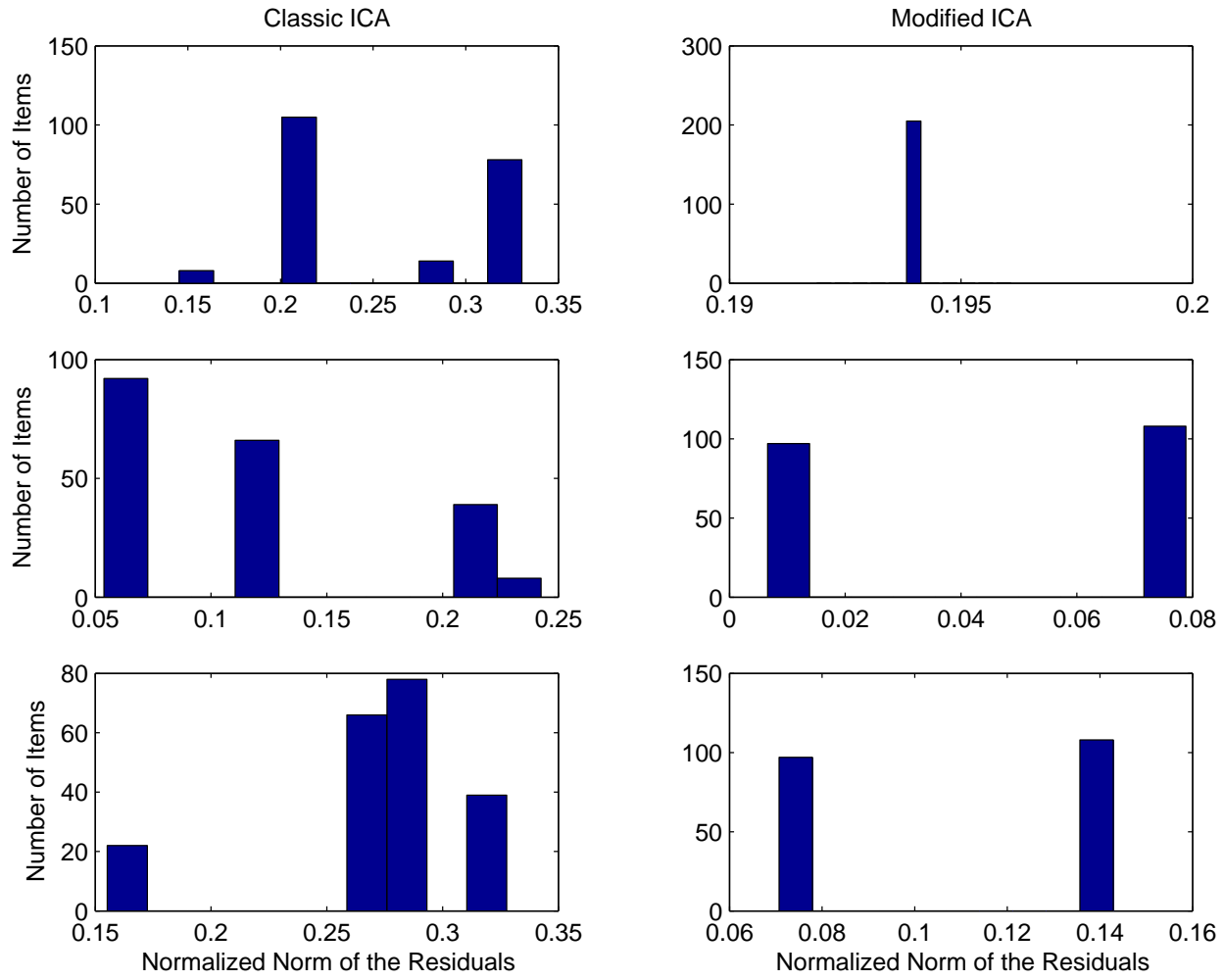
**Fig. 3.** Histogram of the normalised residuals  $\|\hat{\mathbf{S}}_i - \mathbf{S}_i\|/\|\mathbf{S}_i\|$ . The residuals are defined as the difference between the real and the estimated solution. The corresponding solutions of the component separations are obtained with 200 different initializations of the classic ICA and subspace based ICA methods when applied to the observed mixtures in Fig. 1: top panels correspond to component  $i = 1$ , central panels to  $i = 2$ , bottom panels to  $i = 3$ . As shown in the top panel the solution in the modified ICA approach for component  $i = 1$  is stable.



**Fig. 4.** **Top panels** – Original images  $\mathbf{S}$  used in the experiment dealing with component separation as described in Sect. 4; **Bottom panels** – Observed mixtures  $\mathbf{Y} = \mathbf{AS}$ . In this experiment the components are partially correlated with the correlation matrix given by Eq. (23).



**Fig. 5. Top panels** – Typical separation obtained with the classic ICA algorithm when applied to the observed mixtures in Fig. 4; **Bottom panels** – Typical separation obtained with the subspace based ICA when applied to the observed mixtures in Fig. 4.



**Fig. 6.** Same as Figure 2 for the simulations shown in Figure 4 and 5.